

A Note On Cover-Free Families

Mehdi Azadimotlagh

Department of Mathematics

Kharazmi University, 50 Taleghani Avenue, 15618, Tehran, Iran

std_m.azadim@khu.ac.ir

Abstract

Let $N((r, w; d), t)$ denote the minimum number of points in a $(r, w; d)$ -cover-free family having t blocks. Hajiabolhassan and Moazami (2012) [6] showed that the Hadamard conjecture is equivalent to confirm $N((1, 1; d), 4d - 1) = 4d - 1$. Hence, it is a challenging and interesting problem to determine the exact value of $N((r, w; d), t)$. In this paper, we determine the exact value of $N((r, w; d), t)$ for every r, w , where $r + w \leq t$ and some d .

Key words: Cover-free families, Biclique covering number

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1 Introduction

A family of sets is called an (r, w) -cover-free family (or (r, w) -CFF) if no intersection of r sets of the family are covered by a union of any other w sets of the family. Cover-free families were first described by Kautz and Singleton (1964) to investigate superimposed binary codes [7]. Erdos et al. [3] introduced the $(1, r)$ -cover-free family as a generalization of Sperner family. Stinson et al. [10] considered cover-free families as group testing. Mitchell and Piper [8] considered a key distribution pattern which appears to be equivalent to the notion of cover-free family. For another application and discussion of cover-free families, (see, for example, [1, 4, 5, 6, 9, 11, 13]). Stinson and Wei [11] have introduced a generalization of cover-free families as follows.

Definition 1. Let d, n, t, r , and w be positive integers and $B = \{B_1, \dots, B_t\}$ be a collection of subsets of a set X , where $|X| = n$. Each element of the collection B is called a block and the elements of X are called points. The pair (X, B) is called an $(r, w; d)$ -CFF(n, t) if for any two sets of indices $L, M \subseteq [t]$ such that $L \cap M = \emptyset$, $|L| = r$, and $|M| = w$, we have

$$|(\bigcap_{l \in L} B_l) \setminus (\bigcup_{m \in M} B_m)| \geq d.$$

Let $N((r, w; d), t)$ denote the minimum number of points of X in an $(r, w; d)$ -CFF having t blocks. ♠

As was shown by Engel [2], determining the optimal value for a cover-free family is NP-hard. Also, Hajiabolhassan and Moazami [6] showed that the existence of Hadamard matrices results from the existence of some cover-free families and vice versa. A Hadamard matrix of order n is an $n \times n$ matrix H with entries $+1$ and -1 , such that $HH^T = nI_n$.

Theorem A. [6] *Let d be a positive integer, then $N((1, 1; d), 4d - 1) = 4d - 1$ if and only if there exists a Hadamard matrix of order $4d$.*

It is proved that if H is a Hadamard matrix of order n , then $n = 1$, $n = 2$, or $n = 4d$ whenever d is a positive integer [12]. It was conjectured by Jacques Hadamard (1893) that there exists a Hadamard matrix of every order $4d$ whenever d is a positive integer. Actually Hajiabolhassan and Moazami showed that the Hadamard conjecture is equivalent to confirm $N((1, 1; d), 4d - 1) = 4d - 1$. Thus the problem of determining the exact value of the parameter $N((r, w; d), t)$, even for special values of r , w , d , and t is a challenging and interesting problem. In this paper, we determine the exact value of $N((r, w; d), t)$ for every r , w , where $r + w \leq t$ and some d .

2 Cover-Free Family

In this section, we restrict our attention to determine the exact value of $N((r, w; d), t)$ for every value of r , w , and t , where $r + w \leq t$, and some special value of d . In this regard, we need to use some notation and theorem as follows. A biclique of G is a complete bipartite subgraph of G . The d -biclique covering (resp. partition) number $bc_d(G)$ (resp. $bp_d(G)$) of a graph G is the minimum number of bicliques of G such that every edge of G belongs to at least (resp. exactly) d of these bicliques. Hajiabolhassan and Moazami [6] showed that the existence of an $(r, w; d)$ -cover-free family is equivalent to the existence of d -biclique cover of bi-intersection graph. The bi-intersection graph $I_t(r, w)$ is a bipartite graph whose vertices are all w - and r -subsets of a t -element set, where a w -subset is adjacent to an r -subset if and only if their intersection is empty.

Theorem B. [6] *Let r , w , d and t , be positive integers, where $t \geq r + w$. It holds that $N((r, w; d), t) = bc_d(I_t(r, w))$.*

Theorem 1. *Let r , w , and t be positive integers, where $t \geq r + w$. Also, assume that the function $\binom{x}{r} \binom{t-x}{w}$ is maximized for $x = t'$. If $d = \binom{t-r-w}{t'-r}$, then*

$$N((r, w; d), t) = bc_d(I_t(r, w)) = bp_d(I_t(r, w)) = \binom{t}{t'}.$$

Proof. Set $t'' = \binom{t}{t'}$. First, we show that $I_t(r, w)$ can be covered by t'' bicliques such that every edge of $I_t(r, w)$ is covered by exactly d bicliques. Denote the vertex set of $I_t(r, w)$ by bipartition (X, Y) in which $X = \binom{[t]}{r}$ and $Y = \binom{[t]}{w}$. Suppose that A is a t' -subset of $[t]$ and A^c is the complement of the set A in $[t]$. Denote the number of these pairs by t'' . Now, for every t' -subset A_j of $[t]$, where $1 \leq j \leq t''$, construct the biclique G_j with the vertex set (X_j, Y_j) , where $X_j = \binom{A_j}{r}$ and $Y_j = \binom{A_j^c}{w}$. Let UV be an arbitrary edge of $I_t(r, w)$, where $|U| = r$ and $|V| = w$. In view of the definition of G_j , UV is covered by every G_j with vertex set (X_j, Y_j) , where U is a vertex of X_j and V is a vertex of Y_j . Thus every edge of $I_t(r, w)$ is covered by at least d bicliques. One can see that

$$\sum_{j=1}^{t''} |E(G_j)| = \binom{t}{t'} \binom{t'}{r} \binom{t-t'}{w} \quad \& \quad |E(I_t(r, w))| = \binom{t}{r} \binom{t-r}{w}.$$

Now, it is simple to check that

$$\sum_{j=1}^{t''} |E(G_j)| = d|E(I_t(r, w))|.$$

Thus every edge of $I_t(r, w)$ is covered by exactly d bicliques. Note that we have actually proved that

$$bp_d(I_t(r, w)) \leq t''. \quad (1)$$

Conversely, one can see that

$$bp_d(I_t(r, w)) \geq bc_d(I_t(r, w)) \geq \frac{d|E(I_t(r, w))|}{B(I_t(r, w))}.$$

Also, In view of the definition of t' , we have

$$\frac{d|E(I_t(r, w))|}{B(I_t(r, w))} = \frac{\binom{t-r-w}{t'-r} \binom{t}{r} \binom{t-r}{w}}{\binom{t'}{r} \binom{t-t'}{w}} = \binom{t}{t'} = t''.$$

Hence,

$$bp_d(I_t(r, w)) \geq bc_d(I_t(r, w)) \geq t''. \quad (2)$$

From (1) and (2), we conclude

$$bp_d(I_t(r, w)) = bc_d(I_t(r, w)) = t''.$$

By Theorem B,

$$N((r, w; d), t) = bc_d(I_t(r, w)),$$

this completes the proof. ■

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